Available online at www.sciencedirect.com
SCIENCE DIRECT®
Physics Letters B 584 (2004) 204–209
PHYSICS LETTERS B
www.elsevier.com/locate/physletb

Fractional angular momentum in non-commutative spaces

Jian-zu Zhang [a,b]

[a] Institute for Theoretical Physics, Box 316, East China University of Science and Technology, Shanghai 200237, PR China
[b] Department of Physics, University of Kaiserslautern, PO Box 3049, D-67653 Kaiserslautern, Germany

Received 29 December 2003; accepted 21 January 2004

Editor: P.V. Landshoff

Abstract

In noncommutative spaces to maintain Bose–Einstein statistics for identical particles at the non-perturbation level described by deformed annihilation-creation operators when the state vector space of identical bosons is constructed by generalizing one-particle quantum mechanics it is explored that the consistent ansatz of commutation relations of phase space variables should simultaneously include space–space non-commutativity and momentum–momentum non-commutativity, and a new type of boson commutation relations at the deformed level is obtained. Consistent perturbation expansions of deformed annihilation-creation operators are obtained. The influence of the new boson commutation relations on dynamics is discussed. The non-perturbation and perturbation property of the orbital angular momentum of two-dimensional system are investigated. Its spectrum possesses fractional eigen values and fractional intervals.

Recently there has been a renewed interest in physics in non-commutative spaces [1–7]. This is motivated by studies of the low energy effective theory of D-brane with a non-zero NS–NS B field background. The effects of non-commutative spaces only appear near the string scale, thus to test the space non-commutativity we need noncommutative quantum field theories. But study at the level of quantum mechanics in non-commutative spaces is meaningful for clarifying some possible phenomenological consequences in solvable models. In literature the perturbation aspects of non-commutative quantum mechanics (NCQM) have been extensively studied [8–22]. The perturbation approach is based on the Weyl–Moyal correspondence [20–22], according to which the usual product of functions should be replaced by the star-product.

In this Letter we are interested in the non-perturbation investigation which may explores some essentially new features of NCQM. Because of the exponential differential factor in the Weyl–Moyal product the non-perturbation treatment is difficulty. A suitable example for the non-perturbation investigation is two-dimensional isotropic harmonic oscillator. This model is exactly soluble, and explored fully, for example, in [10,13,16]. In the following through the non-perturbation investigation of this example we clarify the consistent condition for quantum theories in non-commutative spaces in detail. The point is how to maintain Bose–Einstein statistics at the non-perturbation level described by deformed annihilation-

E-mail addresses: jzzhang@ecust.edu.cn, jzzahng@physik.uni-kl.de (J.-z. Zhang).

0370-2693 © 2004 Elsevier B.V. Open access under CC BY license.
doi:10.1016/j.physletb.2004.01.049

creation operators in non-commutative spaces when the state vector space of identical bosons is constructed by generalizing one-particle quantum mechanics. We find that for this purpose the consistent ansatz of commutation relations of phase space variables should simultaneously include space–space non-commutativity and momentum–momentum non-commutativity, and a new type of boson commutation relations at the deformed level is obtained. To the linear terms of undeformed annihilation-creation operators, perturbation expansions of deformed annihilation-creation operators are obtained. The influence of the new deformed boson commutation relations on dynamics is investigated. As an example, we study the angular momentum, and explore that its spectrum possesses fractional eigen values and fractional intervals, specially there is a fractional zero-point angular momentum. The perturbation treatment of the harmonic oscillator is briefly presented.

In order to develop the NCQM formulation we need to specify the phase space and the Hilbert space on which operators act. The Hilbert space can consistently be taken to be exactly the same as the Hilbert space of the corresponding commutative system [8].

As for the phase space we consider both space–space non-commutativity (space–time non-commutativity is not considered) and momentum–momentum non-commutativity. There are different types of non-commutative theories, for example, see a review paper [23].

The former is inferred from the string theory [3,4]. The reasons of considering momentum–momentum non-commutativity are: (i) To maintain Bose–Einstein statistics for identical particles at the level of the deformed annihilation-creation operators when the state vector space of identical bosons is constructed by generalizing one-particle quantum mechanics; (ii) To incorporate an additional background magnetic field [10,23]. The points (1) will become clear latter.

In the case of simultaneously space–space non-commutativity and momentum–momentum non-commutativity the consistent NCQM algebra is as follows (henceforth, c and $\hbar$ are set to unit):

$$[\hat{x}_i, \hat{x}_j] = i\xi^{-2}\Lambda_{\mathrm{NC}}^{-2} d\epsilon_{ij}, \qquad [\hat{x}_i, \hat{p}_j] = i\delta_{ij},$$
$$[\hat{p}_i, \hat{p}_j] = i\xi^{-2}\Lambda_{\mathrm{NC}}^{2} d'\epsilon_{ij}, \quad i, j = 1, 2, \tag{1}$$

where d and d' are the constant, frame-independent dimensionless parameters, ϵ_{ij} is a antisymmetric unit tensor, $\epsilon_{12} = -\epsilon_{21} = 1$, $\epsilon_{11} = \epsilon_{22} = 0$. The parameter Λ_{NC} is the NC energy scale [24] (we reset $\theta_{ij} = \Lambda_{\mathrm{NC}}^{-2} d\epsilon_{ij}$ where θ_{ij} is the non-commutative parameter extensively adopted in literature for the case that only space–space are non-commuting). The NC effects will only become apparent as this scale is approached. In Eq. (1) ξ is the scaling factor $\xi = (1 + dd'/4)^{1/2}$. When $d' = 0$, we have $\xi = 1$, the NCQM algebra (1) reduces to the one which is extensively discussed in literature for the case that only space–space are non-commuting. Here and in the following $\hat{F}$ represents the operator in non-commutative spaces, and F the corresponding one in commutative spaces.

The NCQM algebra (1) changes the algebra of creation-annihilation operators. There are different ways to construct the creation-annihilation operators. In order to explore non-commutative effects at the non-perturbation level the deformed creation-annihilation operators are constructed which are related to the non-commutative variables $\hat{x}_i$, $\hat{p}_i$.

The Hamiltonian for two-dimensional isotropic harmonic oscillator is (henceforth, summation convention is used)

$$\hat{H}(\hat{x}, \hat{p}) = \frac{1}{2\mu}\hat{p}_i\hat{p}_i + \frac{1}{2}\mu\omega^2\hat{x}_i\hat{x}_i. \tag{2}$$

This system can be decomposed into two one-dimensional oscillators. For the dimension i the representations of the deformed annihilation-creation operators $\hat{a}_i$, $\hat{a}_i^\dagger$ $(i = 1, 2)$ are defined by

$$\hat{a}_i = \sqrt{\frac{\mu\omega}{2}}\left(\hat{x}_i + \frac{i}{\mu\omega}\hat{p}_i\right),$$
$$\hat{a}_i^\dagger = \sqrt{\frac{\mu\omega}{2}}\left(\hat{x}_i - \frac{i}{\mu\omega}\hat{p}_i\right). \tag{3}$$

In order to maintain the physical meaning of $\hat{a}_i$ and $\hat{a}_i^\dagger$ the relations among $(\hat{a}_i, \hat{a}_i^\dagger)$ and $(\hat{x}_i, \hat{p}_i)$ should keep the same formulation as the ones in commutative spaces.

From Eq. (3) and the NCQM algebra (1) we obtain the commutator between the operators $\hat{a}_i^\dagger$ and $\hat{a}_j^\dagger$: $[\hat{a}_i^\dagger, \hat{a}_j^\dagger] = \frac{i}{2}\xi^{-2}\mu\omega(\Lambda_{\mathrm{NC}}^{-2} d - \mu^{-2}\omega^{-2}\Lambda_{\mathrm{NC}}^{2} d')\epsilon_{ij} \equiv D_{ij}$. If momentum–momentum is commuting, $d' = 0$, for the case $i \neq j$ we would have $D_{ij} \neq 0$ which would lead to that $\hat{a}_i^\dagger$ and $\hat{a}_j^\dagger$ would not commute.

When the state vector space of identical bosons is constructed by generalizing one-particle quantum mechanics, a natural requirement is that Bose–Einstein statistics should be maintained at the non-perturbation level described by $\hat{a}_i^\dagger$. If D_{ij} were not zero, when the operators $\hat{a}_1^\dagger\hat{a}_2^\dagger$ and $\hat{a}_2^\dagger\hat{a}_1^\dagger$ were applied successively to a state, say vacuum state $|0,0\rangle$,[1] would not produce the same physical state: $\hat{a}_1^\dagger\hat{a}_2^\dagger|0,0\rangle - \hat{a}_2^\dagger\hat{a}_1^\dagger|0,0\rangle = D_{ij}|0,0\rangle \neq 0$. In order to maintain Bose–Einstein statistics at the deformed level the basic assumption is that operators $\hat{a}_i^\dagger$ and $\hat{a}_j^\dagger$ are commuting, that is, $D_{ij} \equiv 0$. This requirement leads to a consistency condition

$$d' = \mu^2\omega^2\Lambda_{\mathrm{NC}}^{-4}d. \tag{4}$$

From Eqs. (1), (3) and (4) it follows that the commutation relations of $\hat{a}_i$ and $\hat{a}_j^\dagger$ read

$$\left[\hat{a}_1, \hat{a}_1^\dagger\right] = \left[\hat{a}_2, \hat{a}_2^\dagger\right] = 1, \qquad [\hat{a}_i, \hat{a}_j] = \left[\hat{a}_i^\dagger, \hat{a}_j^\dagger\right] = 0, \quad i, j = 1, 2, \tag{5}$$

$$\left[\hat{a}_1, \hat{a}_2^\dagger\right] = i\xi^{-2}\mu\omega\Lambda_{\mathrm{NC}}^{-2}d. \tag{6}$$

Eq. (5) is the same commutation relations as the ones in the commutative spaces. This confirms that for the same degree of freedom i the operators $\hat{a}_i, \hat{a}_i^\dagger$ are the correct deformed annihilation-creation operators. For the different degrees of freedom new deformed commutation relations (6) between $\hat{a}_i$ and $\hat{a}_j^\dagger$ emerge.

We emphasize that Eq. (6) is consistent with *all* principles of quantum mechanics and Bose–Einstein statistics.[2]

If momentum–momentum is commuting, $d' = 0$, which shows that we could not obtain $D_{ij} = 0$. Thus it is clear that in order to maintain Bose–Einstein statistics for identical bosons at the deformed level of $\hat{a}_i$ and $\hat{a}_i^\dagger$ we should consider both space–space non-commutativity and momentum–momentum non-commutativity.

Now we consider perturbation expansions of $(\hat{x}_i, \hat{p}_j)$ and $(\hat{a}_i, \hat{a}_j^\dagger)$. The NCQM algebra (1) has different possible perturbation realizations [10]. To the linear terms of phase space variables in commutative spaces, a consistency ansatz of the perturbation expansions of $\hat{x}_i$ and $\hat{p}_i$ is

$$\hat{x}_i = \xi^{-1}\left(x_i - \frac{1}{2}\Lambda_{\mathrm{NC}}^{-2}d\epsilon_{ij}p_j\right),$$
$$\hat{p}_i = \xi^{-1}\left(p_i + \frac{1}{2}\mu^2\omega^2\Lambda_{\mathrm{NC}}^{-2}d\epsilon_{ij}x_j\right), \tag{7}$$

where x_i and p_i are the coordinate and momentum in commutative spaces and satisfy the commutation relations $[x_i, x_j] = [p_i, p_j] = 0$, $[x_i, p_j] = i\delta_{ij}$. In commutative spaces the undeformed annihilation-creation operators $(a_i, a_i^\dagger)$ are related to the variables (x_i, p_i) by

$$x_i = \sqrt{\frac{1}{2\mu\omega}}\left(a_i + a_i^\dagger\right), \qquad p_i = \frac{1}{i}\sqrt{\frac{\mu\omega}{2}}\left(a_i - a_i^\dagger\right), \tag{8}$$

where a_i and $a_i^\dagger$ satisfy commutation relations $[a_i, a_j] = [a_i^\dagger, a_j^\dagger] = 0$, $[a_i, a_j^\dagger] = i\delta_{ij}$. Inserting Eqs. (7) and (8) into Eq. (3), to the linear terms of a_i and $a_i^\dagger$, we obtain the perturbation expansions of $\hat{a}_i$ and $\hat{a}_i^\dagger$ as follows

$$\hat{a}_i = \xi^{-1}\left(a_i + \frac{i}{2}\mu\omega\Lambda_{\mathrm{NC}}^{-2}d\epsilon_{ij}a_j\right),$$
$$\hat{a}_i^\dagger = \xi^{-1}\left(a_i^\dagger - \frac{i}{2}\mu\omega\Lambda_{\mathrm{NC}}^{-2}d\epsilon_{ij}a_j\right). \tag{9}$$

It is easy to check that Eqs. (1), (3)–(9) are consistent each other.

Comparing to the case in commutative spaces the deformed commutation relations (6) change dynamical properties of quantum theories in non-commutative spaces. As an example, in the following we investigate the influence of Eq. (6) on the angular momentum. In

---

[1] As in the case of commutative spaces the vacuum state $|0,0\rangle$ in non-commutative spaces is defined as $\hat{a}_i|0,0\rangle = 0$, $i = 1, 2$.

[2] As a check on the consistency, we apply Eq. (6) to the vacuum state $|0,0\rangle$. Because of $\hat{a}_i|0,0\rangle = 0$, the left-hand side reads $(\hat{a}_1\hat{a}_2^\dagger - \hat{a}_2^\dagger\hat{a}_1)|0,0\rangle = \hat{a}_1\hat{a}_2^\dagger|0,0\rangle$. In view of the non-commutativity between $\hat{a}_1$ and $\hat{a}_2^\dagger$, the term $\hat{a}_1\hat{a}_2^\dagger|0,0\rangle \neq 0$. The correct result can be obtained by using the perturbation expansion (9) of $\hat{a}_i$ and $\hat{a}_i^\dagger$:

$$\begin{aligned}\hat{a}_1\hat{a}_2^\dagger|0,0\rangle &= \xi^{-2}\biggl(a_1a_2^\dagger + \frac{i}{2}\mu\omega\Lambda_{\mathrm{NC}}^{-2}da_1a_1^\dagger + \frac{i}{2}\mu\omega\Lambda_{\mathrm{NC}}^{-2}da_2a_2^\dagger \\ &\quad - \frac{1}{4}\mu^2\omega^2\Lambda_{\mathrm{NC}}^{-4}d^2\,a_2a_1^\dagger\biggr)|0,0\rangle \\ &= i\xi^{-2}\mu\omega\Lambda_{\mathrm{NC}}^{-2}d|0,0\rangle\end{aligned}$$

which equals to the right-hand side. (In the last step the undeformed relations $a_ia_j^\dagger - a_j^\dagger a_i = \delta_{ij}$, $a_i|0,0\rangle = 0$ and $a_ia_j^\dagger|0,0\rangle = \delta_{ij}|0,0\rangle$ are used.)

two-dimensions the orbital angular momentum is defined as an exterior product,[3]

$$\hat{L} = \epsilon_{ij}\hat{x}_i\hat{p}_j. \tag{10}$$

Though $\hat{L}$ defined in Eq. (10) has the same formulation as the one in commutative spaces, because of the commutation relations (6) new features appear in its spectrum.

Using Eq. (3) we rewrite $\hat{L}$ as

$$\hat{L} = -i\left(\hat{a}_1^\dagger\hat{a}_2 - \hat{a}_2^\dagger\hat{a}_1\right) - L_0,$$
$$L_0 = \xi^{-2}\mu\omega\Lambda_{\mathrm{NC}}^{-2}d. \tag{11}$$

Where the zero-point angular momentum $L_0 = \langle 0, 0|\hat{L}|0,0\rangle$ originates from the deformed commutation relations (6).

In order to clarify the origin of the zero point angular momentum L_0 from another point of view, in the following we investigate the perturbation expansion of $\hat{L}$. Starting from Eq. (10), using Eqs. (7) we obtain

$$\hat{L} = \epsilon_{ij}x_ip_j - \xi^{-2}\mu\Lambda_{\mathrm{NC}}^{-2}d\left(\frac{1}{2\mu}p_ip_i + \frac{1}{2}\mu\omega^2x_ix_i\right). \tag{12}$$

Eq. (12) is exactly solvable. We change the variables x_i and p_i to variables X_α and P_α (here and in the following $\alpha, \beta = a, b$) as follows [26,27],

$$X_a = \sqrt{\frac{\mu\omega}{2\Omega_a}}\,x_1 - \sqrt{\frac{1}{2\mu\omega\Omega_a}}\,p_2,$$
$$X_b = \sqrt{\frac{\mu\omega}{2\Omega_b}}\,x_1 + \sqrt{\frac{1}{2\mu\omega\Omega_b}}\,p_2,$$
$$P_a = \sqrt{\frac{\Omega_a}{2\mu\omega}}\,p_1 + \sqrt{\frac{\mu\omega\Omega_a}{2}}\,x_2,$$
$$P_b = \sqrt{\frac{\Omega_b}{2\mu\omega}}\,p_1 - \sqrt{\frac{\mu\omega\Omega_b}{2}}\,x_2, \tag{13}$$

where

$$\Omega_a = \omega(1+L_0), \qquad \Omega_b = \omega(1-L_0). \tag{14}$$

In the above X_α and P_α satisfy $X_\alpha = X_\alpha^\dagger$, $P_\alpha = P_\alpha^\dagger$, $[X_\alpha, X_\beta] = [P_\alpha, P_\beta] = 0$ and $[X_\alpha, P_\beta] = i\delta_{\alpha\beta}$. Then we define following annihilation-creation operators $A_\alpha, A_\alpha^\dagger$

$$A_\alpha = i\sqrt{\frac{1}{2\Omega_\alpha}}\,P_\alpha + \sqrt{\frac{\Omega_\alpha}{2}}\,X_\alpha,$$
$$A_\alpha^\dagger = -i\sqrt{\frac{1}{2\Omega_\alpha}}\,P_\alpha + \sqrt{\frac{\Omega_\alpha}{2}}\,X_\alpha, \quad \alpha = a, b. \tag{15}$$

Here A_α and $A_\alpha^\dagger$ satisfy $[A_\alpha, A_\beta] = [A_\alpha^\dagger, A_\beta^\dagger] = 0$, and $[A_\alpha, A_\beta^\dagger] = \delta_{\alpha\beta}$. The eigenvalues n_α of the number operator $N_\alpha = A_\alpha^\dagger A_\alpha$ are $n_\alpha = 0, 1, 2, \ldots$. The angular momentum (12) is rewritten in the form of two uncoupled modes of frequencies Ω_a and Ω_b:

$$\hat{L} = \frac{\Omega_b}{\omega}A_b^\dagger A_b - \frac{\Omega_a}{\omega}A_a^\dagger A_a - L_0. \tag{16}$$

Where the zero point angular momentum L_0 appears again. The spectrum of $\hat{L}$ is

$$l_{(n_a,n_b)} = \frac{\Omega_b}{\omega}n_b - \frac{\Omega_a}{\omega}n_a - L_0$$
$$= n_b - n_a - (n_a + n_b + 1)L_0. \tag{17}$$

$l_{(n_a,n_b)}$ can take integers or fractional values. We notice that, unlike the case in commutative spaces, the absolute value of the lowest angular momentum is not zero, it is $|L_0| = \xi^{-2}\mu\omega\Lambda_{\mathrm{NC}}^{-2}|d|$. For the case $n_a = n_b = n$ the eigenvalues are proportional to L_0

$$l_{(n,n)} = -(2n+1)L_0 \tag{18}$$

which are fractional values. For the case $n_b = -(n_a + 1) = n$ the eigenvalues are

$$l_{(-n-1,n)} = 2n+1 \tag{19}$$

which are integer. From Eq. (17) the intervals of the spectrum are $\Delta l_{(\Delta n_a, \Delta n_b)} = l_{(n'_a,n'_b)} - l_{(n_a,n_b)} = \Delta n_b - \Delta n_a - (\Delta n_a + \Delta n_b)L_0$ where $\Delta n_a = n'_a -$

[3] There are different ways to define the angular momentum in non-commutative spaces. In [10] comparing to the case in commutative spaces, the angular momentum acquires d- and d'-dependent scalar terms $\hat{x}_i\hat{x}_i$ and $\hat{p}_i\hat{p}_i$. Because of the scalar terms have nothing to do with the angular momentum, so in this Letter we prefer to keep the same definition of $\hat{L}$ as the one in commutative spaces. Eq. (6) modifies the commutation relations between $\hat{L}$ and $\hat{x}_i$, $\hat{p}_i$. From the NCQM algebra (1) we obtain

$$[\hat{L}, \hat{x}_i] = i\epsilon_{ij}\hat{x}_j + i\xi^{-2}\Lambda_{\mathrm{NC}}^{-2}d\,\hat{p}_i,$$
$$[\hat{L}, \hat{p}_i] = i\epsilon_{ij}\hat{p}_j - i\xi^{-2}\mu^2\omega^2\Lambda_{\mathrm{NC}}^{-2}d\,\hat{x}_i.$$

Comparing to the commutative case, the above commutation relations acquires d- and d'-dependent terms which represent effects in non-commutative spaces.

n_a and $\Delta n_b = n'_b - n_b$. There are different intervals in the spectrum. For the case $\Delta n_a = \Delta n_b = \Delta n$ we have fractional intervals $\Delta l_{(\Delta n,\Delta n)} = -2\Delta n L_0$. For the case $\Delta n_b = -\Delta n_a = \Delta n$ we have integer intervals $\Delta l_{(-\Delta n,\Delta n)} = 2\Delta n$. We notice that the zero-point value L_0, the spectrum $l_{(n,n)}$ and the intervals $\Delta l_{(\Delta n,\Delta n)}$ take fractional values which are parameter d-dependent. Such fractional feature of the orbital angular momentum represents the effects of non-commutative spaces.

In the limit $d \to 0$ we have $L_0 \to 0$, $\omega_a, \omega_b \to \omega$. Thus the spectrum $l_{(n_a,n_b)} \to \omega(n_b - n_a)$ which recovers the results in commutative spaces.

For consistent check, we use Eq. (9) to investigate the perturbation expansion of the first term in Eq. (11). We obtain

$$-i\left(\hat{a}_1^\dagger \hat{a}_2 - \hat{a}_2^\dagger \hat{a}_1\right) = -i\left(a_1^\dagger a_2 - a_2^\dagger a_1\right) - \xi^{-2}\mu\omega\Lambda_{\mathrm{NC}}^{-2} d\left(a_1^\dagger a_1 + a_2^\dagger a_2\right).$$

The operators a_1 and a_2 are related to the operators A_a and A_b by the following equations

$$A_a = \frac{1}{\sqrt{2}}(a_1 + ia_2), \qquad A_b = \frac{1}{\sqrt{2}}(a_1 - ia_2). \tag{20}$$

It follows that $-i(\hat{a}_1^\dagger \hat{a}_2 - \hat{a}_2^\dagger \hat{a}_1) = \frac{\Omega_b}{\omega} A_b^\dagger A_b - \frac{\Omega_a}{\omega} A_a^\dagger A_a$. Thus Eq. (11) equals Eq. (16).

To conclude this paper we investigate the perturbation treatment of the harmonic oscillator. Using Eq. (7), the perturbation expansion of Eq. (2) is

$$\hat{H}(\hat{x}, \hat{p}) = \frac{1}{2\mu} p_i p_i + \frac{1}{2}\mu\omega^2 x_i x_i - \xi^{-2}\mu\omega^2 \Lambda_{\mathrm{NC}}^{-2} d\epsilon_{ij} x_i p_j. \tag{21}$$

In the above the Chern–Simons term $-\xi^{-2}\mu\omega^2\Lambda_{\mathrm{NC}}^{-2} \times d\epsilon_{ij} x_i p_j$ represents the effect of non-commutative spaces. Because of the scaling factor ξ in Eqs. (7) the corrections from the kinetic energy term and the potential term cancel each other, thus in Eq. (21) there are no corrections to the mass and frequency. These are different from the results obtained in [25] where the effective mass and frequency have d-dependent corrections.

Eq. (21) is exactly solvable. By the same procedure of obtaining (16) we rewrite the Hamiltonian (21) as

$$H = H_a + H_b, \qquad H_\alpha = \Omega_\alpha\left(A_\alpha^\dagger A_\alpha + \frac{1}{2}\right), \quad \alpha = a, b. \tag{22}$$

Because of $\Omega_a \neq \Omega_b$ the eigenvalues of H are non-degenerate.

In summary, in non-commutative spaces the maintenance of Bose–Einstein statistics at the deformed level of $\hat{a}_i$ and $\hat{a}_i^\dagger$ requires *simultaneously* space–space non-commutativity and momentum–momentum noncommutativity. The commutation relations (5) and (6), and the perturbation expansions (7) and (9) are general, they provide a consistent framework for the further development of quantum theories in non-commutative spaces. Further exploration of the influence of the new deformed commutation relations (6) on dynamics is interesting.

Acknowledgements

This work has been supported by the Deutsche Forschungsgemeinschaft (Germany). The author would like to thank W. Rühl for stimulating discussions. His work has also been supported by the National Natural Science Foundation of China under the grant number 10074014 and by the Shanghai Education Development Foundation.

References

[1] A. Connes, M.R. Douglas, A. Schwarz, JHEP 9802 (1998) 003, hep-th/9711162.
[2] M.R. Douglas, C.M. Hull, JHEP 9802 (1998) 008, hep-th/9711165.
[3] F. Ardalan, H. Arfaei, M.M. Sheikh-Jabbari, JHEP 9902 (1999) 016, hep-th/9810072.
[4] S.-C. Chu, P.-M. Ho, Nucl. Phys. B 550 (1999) 151, hep-th/9812219.
[5] S.-C. Chu, P.-M. Ho, Nucl. Phys. B 568 (2000) 447, hep-th/9906192.
[6] V. Schomerus, JHEP 9906 (1999) 030, hep-th/9903205.
[7] N. Seiberg, E. Witten, JHEP 9909 (1999) 032, hep-th/9908142.
[8] M. Chaichian, M.M. Sheikh-Jabbari, A. Tureanu, Phys. Rev. Lett. 86 (2001) 2716, hep-th/0010175.
[9] J. Gamboa, M. Loewe, J.C. Rojas, Phys. Rev. D 64 (2001) 067901, hep-th/0010220.
[10] V.P. Nair, A.P. Polychronakos, Phys. Lett. B 505 (2001) 267, hep-th/0011172.
[11] D. Kochan, M. Demetrian, hep-th/0102050.
[12] B. Morariu, A.P. Polychronakos, Nucl. Phys. B 610 (2001) 531, hep-th/0102157.
[13] A. Hatzinikitas, I. Smyrnakis, J. Math. Phys. 43 (2002) 113, hep-th/0103074.
[14] J. Gamboa, M. Loewe, F. Mendez, J.C. Rojas, Int. J. Mod. Phys. A 17 (1999) 2555, hep-th/0106125.

[15] S. Bellucci, A. Neressian, C. Sochichiu, Phys. Lett. B 522 (2001) 345, hep-th/0106138.
[16] A. Smailagic, E. Spallucci, Phys. Rev. D 65 (2002) 107701, hep-th/0108216.
[17] H.R. Christiansen, F.A. Schaposnik, Phys. Rev. D 65 (2002) 086005, hep-th/0106181.
[18] C. Acatrinei, JHEP 0109 (2001) 007, hep-th/0107078.
[19] P.-M. Ho, H.-C. Kao, Phys. Rev. Lett. 88 (2002) 151602, hep-th/0110191.
[20] L. Alvarez-Gaume, S.R. Wadia, Phys. Lett. B 501 (2001) 319, hep-th/0006219.
[21] A. Micu, M.M. Sheikh-Jabbari, JHEP 0101 (2001) 025, hep-th/0008057.
[22] I.F. Riad, M.M. Sheikh-Jabbari, JHEP 0008 (2000) 045, hep-th/0008132.
[23] M.R. Douglas, N.A. Nekrasov, Rev. Mod. Phys. 73 (2001) 977, hep-th/0106048.
[24] If the Planck scale is indeed of order a few TeV, the NC energy scale is anticipated to lie above a TeV, for example, see J.L. Hewett, F.J. Petriello, T.G. Rizzo, Phys. Rev. D 64 (2000) 075012, hep-ph/0010354;
J.L. Hewett, F.J. Petriello, T.G. Rizzo, hep-ph/0201275.
[25] B. Muthukumar, P. Mitra, Phys. Rev. D 66 (2002) 027701, hep-ph/0204149.
[26] C. Baxter, Phys. Rev. Lett. 74 (1995) 514.
[27] J.-z. Zhang, Phys. Rev. Lett. 77 (1996) 44.